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HUYGENS' PRINCIPLE AND SEMILINEAR WAVE EQUATIONS

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Abstract—We prove that there are many C^∞ solutions of the semilinear wave equations $u_{tt} - \Delta u + f(u) = 0$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, with C^∞ initial data and compact support, with the property that they do not propagate on spherical shells. Our method is elementary and works for a large class of nonlinearities, which includes the case $f(s) = s^3$.

1. INTRODUCTION

Our aim in this work will be to answer the following question: suppose we consider global smooth solutions of semilinear wave equations in $\mathbb{R}^3$ whose initial data have compact support. Do they propagate on spherical shells? Or equivalently: does Huygens' principle hold for semilinear wave equations in three spatial dimensions?

The classical (strong) Huygens' principle, which is valid for the free wave equation $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times \mathbb{R}$ ($n = \text{odd} \geqslant 3$) says that if u is a C^∞ solution with initial data supported by the ball $\{|x| \leqslant R\}$ then $u \equiv 0$ in the double cone

$$\{(x, t), |x| \leqslant |t| - R, |t| \geqslant R\}. \tag{1}$$

Our main result in this paper implies that the above property ceases to hold (in general) for C^∞ solutions of

$$u_{tt} - \Delta u + u^3 = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}. \tag{2}$$

In fact, we shall prove (Theorem 3.2) that we can find a C^∞ solution of equation (2) with C^∞ initial data supported in a ball $\{|x| \leqslant R\}$, which does not vanish identically in the double cone described in property (1).

Some caution may be necessary before discarding the possibility that Huygens' principle may be valid for general nonlinear wave equations as the following example due to *L.* Niremberg shows: let g: $\mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $g' \neq 0$. Let u: $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be also a function of class C^∞. The substitution $v = g(u)$ and a straightforward calculation give us the relation

$$v_{tt} - \Delta v = g'(u)[u_{tt} - \Delta u] + g''(u)[u_t^2 - |\nabla u|^2] \tag{3}$$

for all $x \in \mathbb{R}^3$, $t \in \mathbb{R}$. Here Δ denotes the Laplacian operator and ∇u is the gradient of u with respect to the spatial variables only. Let $g(s) = 1 - \exp(s)$ and consider the Cauchy problem

$$\begin{gathered} u_{tt} - \Delta u + u_t^2 - |\nabla u|^2 = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R} \\ u(x, 0) = h_1(x), \quad u_t(x, 0) = h_2(x) \end{gathered} \tag{4}$$

where h_1 and h_2 are $C^\infty(\mathbb{R}^3)$ functions with compact support. Then, relation (3) implies that u enjoys Huygens' principle because v does.

There is an extensive literature on the question of the validity of Huygens' principle for second order linear partial differential equations of normal hyperbolic type. We refer to the recent survey article of McLenaghan [1] and the references included there. Although it is intuitively expected, as far as we know, the nonlinear case has not appeared in the literature.

Let us describe briefly the context of this paper: after some technical lemmas in Section 2, we study in Section 3 the "purely" power case, i.e. equation (2), because this is one of the most frequent nonlinearities which appears in the applications. In Section 4 we present some simple generalizations of our techniques.

In what follows we shall use standard notation: by $L^p(\mathbb{R}^3)$ we shall denote the space of (classes of) functions in $\mathbb{R}^3$ whose pth power is integrable with the norm

$$\|g\|_p^p = \int |g(x)|^p \, \mathrm{d}x, \quad (1 \leqslant p < \infty).$$

By $L^\infty(\mathbb{R}^3)$ we denote the space of measurable essentially bounded functions in $\mathbb{R}^3$ with the norm $\|g\|_\infty = \operatorname{ess\,sup}|g(x)|$. An integral sign to which no domain is attached will be understood to be taken over all $\mathbb{R}^3$. By $W^{k,s}(\mathbb{R}^3)$ we denote the Sobolev space of (classes of) functions in $L^s(\mathbb{R}^3)$ which together with their partial derivatives (in the sense of distributions) up to order k belong to $L^s(\mathbb{R}^3)$. By ∇u we mean the gradient of u (in space variables) and

$$|\nabla u|^2 = \sum_{j=1}^{3} \left(\frac{\partial u}{\partial x_j}\right)^2.$$

Δ denotes the Laplacian operator, i.e.

$$\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}.$$

The expression $\mathrm{d}Sy$ will always mean the surface measure with respect to the variable y. Since equation (2) is reversible in time, we shall perform our estimates only for $t > 0$ and the same will be true for $t < 0$. All functions considered in this paper are real-valued. We denote by $C_0^\infty(\mathbb{R}^3)$ the space of all C^∞ functions defined on $\mathbb{R}^3$ with real values and compact support. If X is a Banach space and $1 \leqslant p \leqslant +\infty$, we denote by $L^p(\mathbb{R}^+; X)$ the space of measurable vector functions u: $\mathbb{R}^+ \to X$ such that $\|u(t)\|_X \in L^p(\mathbb{R}^+)$.

2. SOME TECHNICAL LEMMAS

Lemma 2.1

Let g: $\mathbb{R}^3 \to \mathbb{R}$ belonging to $W^{1,p}(\mathbb{R}^3)$ with $1 \leqslant p < 3$. Then, for any positive t we have

$$\int_{|y|=t} |g(y)| \, \mathrm{d}Sy \leqslant c(p) t^{3(p-1)/p} \|\nabla g\|_p$$

where

$$c(p) = \begin{cases} [4\pi(p-1)/3-p]^{p-1/p}, & \text{if } 1 < p < 3 \\ 1, & \text{if } p = 1 \end{cases}. \tag{5}$$

Proof. If suffices to prove the lemma for $g \in S$, i.e. the Schwartz's space of rapidly decreasing functions. For any $y \in \mathbb{R}^3 - \{0\}$ and $g \in S$, we have

$$g(y) = \int_1^\infty \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda y) \, \mathrm{d}\lambda = \int_1^\infty \nabla g(\lambda y) \cdot y \, \mathrm{d}\lambda$$

where the dot $\cdot$ means inner product in $\mathbb{R}^3$. Thus

$$\int_{|y|=t} |g(y)| \, \mathrm{d}Sy \leqslant t \int_{|y|=t} \int_1^\infty |\nabla g(\lambda y)| \, \mathrm{d}\lambda \, \mathrm{d}Sy = t \int_{|y|=t} \int_1^\infty |\nabla g(\lambda y)| \lambda^{2/p} \lambda^{-2/p} \, \mathrm{d}\lambda \, \mathrm{d}Sy. \tag{6}$$

Using Hölder's inequality in expression (6) (when $1 < p < 3$) it follows that

$$\int_{|g|=t} |g(y)| \, \mathrm{d}Sy \leqslant c(p) t^{3p-2/p} \left[\int_{|y|=t} \int_1^\infty |\nabla g(\lambda y)|^p \lambda^2 \, \mathrm{d}\lambda \, \mathrm{d}Sy \right]^{1/p} \tag{7}$$

where $c(p)$ is given by conditions (5). The case $p = 1$ follows directly from expression (6). Finally, we observe that

$$\int_{|y|=t}\int_1^\infty |\nabla g(\lambda y)|^p \lambda^2\, d\lambda\, dSy = t^{-1}\int_{|w|=1}\int_t^\infty |\nabla g(\theta w)|^p \theta^2\, d\theta\, dS_w \leqslant t^{-1}\|\nabla g\|_p^p. \tag{8}$$

Combining expressions (8) with (3) we conclude the proof of lemma 2.1. □

Next, we consider continuous functions F: $\mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$. For such F's let L the integral operator defined by

$$(LF)(x, t) = (4\pi)^{-1}\int_0^t \int_{|y|=t-s} f(x+y, s)(t-s)^{-1}\, dSy\, ds. \tag{9}$$

We observe that the operator L is "essentially" the inverse of the d'Alembertian operator $\partial^2/\partial t^2 - \Delta$. In fact, a straightforward calculation shows that if $F = F(x, t)$ is a function of class C^2, then $LF(x, t)$ is of class C^2 and $(LF)_{tt} - \Delta(LF) = F$ for $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ satisfying $LF(x, 0) = 0$, $(\partial/\partial t)LF(x, 0) = 0$.

The following lemma holds in more general circumstances (see the chain rule in $W^{k,s}$ [2]) but for our purposes we only need the simple case of Lemma 2.2.

Lemma 2.2

Let u: $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be such that for each t, $u(\cdot, t) \in W^{1,2}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ and let f: $\mathbb{R} \to \mathbb{R}$, $f \in C^1$ such that for each t, $(f \circ u)(\cdot, t) \in W^{1,p}(\mathbb{R}^3)$.

Furthermore, we assume that $\nabla u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))$ and $f'(u) \in L^\infty(\mathbb{R}^+; L^{2p/2-p}(\mathbb{R}^3))$, where $1 < p \leqslant 2$. Then for $x \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$ we have the estimate

$$|Lf(u)(x, t)| \leqslant pc(p)[12\pi(p-1)]^{-1} t^{3(p-1)/p} \sup_{0 \leqslant s \leqslant t} \|\nabla u(\cdot, s)\|_2 \|f'(u)\|_{2p/2-p}$$

where $c(p)$ and L are given by conditions (5) and (9), respectively.

Proof. Since $(f \circ u)(\cdot, t) \in W^{1,p}(\mathbb{R}^3)$ we can apply Lemma 2.1 to obtain

$$|Lf(u)(x, t)| \leqslant c(p)(4\pi)^{-1}\int_0^t (t-s)^{2-3/p}\|\nabla f(u(\cdot, s))\|_p\, ds. \tag{10}$$

Using the Chain rule and Hölder's inequality it follows that

$$\|\nabla f(u)\|_p = \|f'(u)\nabla u\|_p \leqslant \|\nabla u\|_2 \|f'(u)\|_{2p/2-p} \tag{11}$$

provided $1 < p \leqslant 2$. Thus, from inequalities (10) and (11) we obtain

$$|Lf(u)(x, t)| \leqslant pc(p)[12\pi(p-1)]^{-1} t^{3(p-1)/p} \sup_{0 \leqslant s \leqslant t} \|\nabla u\|_2 \|f'(u)\|_{2p/2-p}$$

which proves Lemma 2.2. □

Corollary 2.1

Let u as in Lemma 2.2. Then for each $t < 0$ and $x \in \mathbb{R}^3$ we have that

$$|L(u^3)(x, t)| \leqslant \tilde{C}\sqrt{t} \sup_{0 \leqslant s \leqslant t} \|\nabla u(\cdot, t)\|_2^3 \tag{12}$$

where $\tilde{C} = 2(4\pi)^{1/6} 9^{-1/6} 3^{-1/2} \pi^{-3} \cong 0.0393$.

Proof. We use Lemma 2.2 with $f(s) = s^3$ and $p = 6/5$. The embedding $W^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ implies that all the other assumptions of Lemma 2.2 hold, thus

$$|L(u^3)(x, t)| \leqslant C\sqrt{t} \sup_{0 \leqslant s \leqslant t} \|u\|_6^2 \|\nabla u\|_2 \tag{13}$$

where $C = 3(2\pi)^{-1}(4\pi)^{1/6} 9^{-1/6}$. Now, we use Sobolev's inequality with the best possible constant, i.e. $\|u\|_6 \leqslant 4(3\sqrt{3}\pi^2)^{-1}\|\nabla u\|_2$ (see Ref. [3]). This, together with expression (13) proves Corollary 2.1. □

3. THE MAIN RESULT

Let $\epsilon > 0$ and $u = u(x, t, \epsilon)$ be the (unique) global C^∞ solution of the Cauchy problem

$$\begin{aligned} &u_{tt} - \Delta u + \epsilon u^3 = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}^+ \\ &u(x, 0) = 0, \quad u_t(x, 0) = h(x) \end{aligned} \tag{14}$$

where $h \in C_0^\infty(\mathbb{R}^3)$.

It is well-known that there exists such a unique global C^∞ solution of problem (14) (see Refs [4, 5]).

Lemma 3.1

Let $u = u(x, t, \epsilon)$ be the solution of problem (14) and L given by equation (9). Then, for any $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ we have

$$|Lu^3(x, t)| \leqslant \tilde{C}\sqrt{t}\,\|h\|_2^3$$

where $\tilde{C}$ is the positive constant in expression (12).

Proof. We consider the energy $E(t)$ given by

$$E(t) = \frac{1}{2}\int (u_t^2 + |\nabla u|^2)\,\mathrm{d}x + \frac{\epsilon}{4}\int u^4\,\mathrm{d}x.$$

It is well-known that $E(t)$ is independent of time. In particular,

$$\tfrac{1}{2}\|\nabla u(\cdot, t)\|_2^2 \leqslant E(t) = E(0) = \tfrac{1}{2}\|h\|_2^2.$$

Since finite propagation speed holds for the solution u of problem (14), then we can apply Corollary 2.1 to conclude the proof. □

Let us denote by $v = v(x, t)$ the solution of the "free" wave equation with the same initial conditions as u in problem (14):

$$\begin{aligned} &v_{tt} - \Delta v = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}^+ \\ &v(x, 0) = 0, \quad v_t(x, 0) = h(x) \end{aligned} \tag{15}$$

Lemma 3.2

Let $\epsilon > 0$, h as in problem (14) and $u = u(x, t, \epsilon)$ be the solution of problem (14). Furthermore, assume that $h \geqslant 0$ and say the support of h is contained in the ball $\{|x| \leqslant R\}$.

Let $(x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}^+$ be such that $t_0 \geqslant R$ and $|x_0| \leqslant t_0 - R$ and L be given by equation (9). If $Lv^3(x_0, t_0) \neq 0$ then there exists $\epsilon_0 = \epsilon_0(t_0) > 0$ such that

$$u(x_0, t_0, \epsilon) \neq 0 \quad \text{for } 0 < \epsilon \leqslant \epsilon_0.$$

Proof. Using the variation of parameters formula we can write problem (14) as

$$u = v - \epsilon L(u^3). \tag{16}$$

Let us iterate equation (16) to obtain

$$\begin{aligned} u &= v - \epsilon L(v - \epsilon Lu^3)^3 \\ &= v - \epsilon Lv^3 + 3\epsilon^2 L(v^2 Lu^3) - 3\epsilon^2 L(v(Lu^3)^2) + \epsilon^4 L(Lu^3)^3. \end{aligned} \tag{17}$$

Let us suppose that there exists a sequence $\{\epsilon_n\}_{n=1}^\infty$ with $\epsilon_n > 0$, $\epsilon_n \to 0$ and $u(x_0, t_0, \epsilon_n) = 0$. Since v verifies Huygens' principle, if we evaluate identity (17) in (x_0, t_0) and take $\epsilon = \epsilon_n$ then it follows that

$$\begin{aligned} Lv^3(x_0, t_0) &= [3\epsilon_n L(v^2 Lu^3) - 3\epsilon_n^2 L(vLu^3)^2 + \epsilon_n^3 L(Lu^3)^2](x_0, t_0) \\ &\leqslant 3\epsilon_n L(v^2 Lu^3)(x_0, t_0) + \epsilon_n^3 L(Lu^3)^3(x_0, t_0) \end{aligned} \tag{18}$$

because $v \geqslant 0$ (for all $x \in \mathbb{R}^3$, $t \geqslant 0$) and L takes non-negative functions into non-negative functions. Now, we use Lemma 3.1 to obtain the estimate

$$|L(v^2 Lu^3)(x_0, t_0)| \leqslant \sup_{(x,t)\in\Omega(x_0,t_0)} |Lu^3(x,t)|Lv^2(x_0,t_0) \leqslant \tilde{C}\sqrt{t_0}\|h\|_2^3 Lv^2(x_0,t_0) \tag{19}$$

where $\Omega(x_0, t_0) = \{(y, s) \in \mathbb{R}^3 \times \mathbb{R}^+, |y - x_0| = t_0 - s, 0 \leqslant s \leqslant t_0\}$. Similarly

$$|L(Lu^3)(x_0, t_0)| \leqslant \sup_{(x,t)\in\Omega(x_0,t_0)} |(Lu^3)^3(x,t)|L(1)(x_0,t_0) \leqslant \tilde{\tilde{C}} = \text{Constant} \tag{20}$$

where $\tilde{\tilde{c}} = \tilde{\tilde{c}}(x_0, t_0, h) > 0$. From expressions (18)–(20) we conclude that

$$|Lv^3(x_0, t_0)| \leqslant \epsilon_n \text{ Constant}. \tag{21}$$

Letting $n \to +\infty$ in expression (21) we deduce that $Lv^3(x_0, t_0) = 0$ which is a contradiction and this proves Lemma 3.2. □

Theorem 3.1

Let h and v be as in Lemma 3.2. Let $\rho \geqslant R$. Then, there exists $\epsilon_1 = \epsilon_1(\rho)$ such that if $0 < \epsilon \leqslant \epsilon_1$, the corresponding solution $u = u(x, t, \epsilon)$ of problem (14) does not vanish (identically) in the forward cone

$$K_\rho = \{(x, t), |x| \leqslant t - \rho, t \geqslant \rho\}.$$

Proof. By Lemma 3.2, it suffices to verify the existence of some $(x_0, t_0) \in K_\rho$ such that $Lv^3(x_0, t_0)$ is different from zero. Actually, for the type of initial data we are considering we can show that the result is true everywhere in K_ρ. In fact let $(x_0, t_0) \in K_\rho$. Let us denote by $G = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+, v(x, t) \neq 0\}$ and $\Omega(x_0, t_0) = \{(y, s) \in \mathbb{R}^3 \times \mathbb{R}^+, |y - x_0| = t_0 - s, 0 \leqslant s \leqslant t_0\}$. Since v is given by

$$v(x, t) = (4\pi)^{-1} t \int_{|y|=1} h(x + ty)\, \mathrm{d}Sy$$

then, clearly $\Omega(x_0, t_0) \cap G$ is not empty. Also, G is an open set in $\mathbb{R}^3 \times \mathbb{R}$, therefore $\Omega(x_0, t_0) \cap G$ is an open set in $\Omega(x_0, t_0)$ in which $v^3 > 0$. Hence $Lv^3(x_0, t_0)$ being an integral over $\Omega(x_0, t_0)$ is positive. □

Theorem 3.2

Let h and v be as in Lemma 3.2. Given any $\rho \geqslant R$, there exists a solution of class C^∞ of

$$w_{tt} - \Delta w + w^3 = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}^+ \tag{22}$$

with initial data supported in $\{|x| \leqslant R\}$ such that w does not vanish (identically) in $K_\rho = \{(x, t), |x| \leqslant t - \rho, t \geqslant \rho\}$.

Proof. It follows as a direct application of Theorem 3.1 and appropriate scaling. In fact, let us choose $u = u(x, t, \epsilon)$ (with $0 < \epsilon \leqslant \epsilon_1$) the solution of problem (14) which does not vanish identically in K_ρ. Let $w = \sqrt{\epsilon} u$, then w satisfies equation (22) with initial data $w(x, 0) = 0$, $w_t(x, 0) = \sqrt{\epsilon} h(x)$ with the requirements of Theorem 3.2. □

4. SOME GENERALIZATIONS

There are well-known results which give us appropriate conditions on the nonlinearity so that we have global smooth solutions of

$$u_{tt} - \Delta u + f(u) = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R} \tag{23}$$

(see for example, Jörgens [4] and Strauss [5]).

Assume the following conditions:

(H1) Let f: $\mathbb{R} \to \mathbb{R}$ be a C^∞ function.

(H2) Let
$$F(s) = \int_0^s f(\lambda)\, \mathrm{d}\lambda.$$

We assume that $F(s) \geqslant 0$ for all s and $|f(s)| = 0(1 + |s|^{1-\epsilon} F(s)^{-\epsilon + 2/3})$ as $|s| \to +\infty$ for some $\epsilon > 0$.

It is well-known [5] that under the above conditions and initial data belonging to $C_0^\infty(\mathbb{R}^3)$ then, we have a unique C^∞ solution of equation (23).

Due to technical reasons we shall assume a stronger condition than the second part of condition (H2):

(H3) There exists α, $0 \leqslant \alpha < 3$ and $C > 0$ such that $|f'(s)| \leqslant C|s|^\alpha$ for all $s \in \mathbb{R}$.

Furthermore:

(H4) $uf(u) > 0$ for all $u \neq 0$.

Lemma 4.1

Assume conditions (H1)–(H4). Let $\epsilon > 0$ and $u = u(x, t, \epsilon)$ be the solution of the Cauchy problem

$$u_{tt} - \Delta u + \epsilon f(u) = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}$$
$$u(x, 0) = 0, \quad u_t(x, 0) = h(x) \tag{24}$$

where $h \in C_0^\infty(\mathbb{R}^3)$, then

$$|Lf(u)(x, t)| \leqslant C(\alpha) t^{(3-\alpha)/2} \| h \|_2^{\alpha+1}$$

for any $x \in \mathbb{R}^3$, $t \geqslant 0$. Here L is given by equation (9) and $C(\alpha)$ is a positive constant which depends only on α.

Proof. It is well-known that the solution of problem (24) enjoys finite speed propagation, therefore in order to apply Lemma 2.2 it remains only to estimate $\| f'(u)(\cdot, t) \|_{2p/2-p}$ for some $1 < p \leqslant 2$. Let us choose $p = 6/(\alpha + 3)$, then using condition (H3), we have

$$\| f'(u)(\cdot, t) \|_{2p/2-p} \leqslant C \| u(\cdot, t) \|_6^\alpha \leqslant \text{Constant} \| \nabla u(\cdot, t) \|_2^\alpha$$

because of Sobolev's inequality. Thus, by Lemma 2.2 we obtain

$$|Lf(u)(x, t)| \leqslant c(\alpha) t^{(3-\alpha)/2} \sup_{0 \leqslant s \leqslant t} \| \nabla u(\cdot, t) \|^{\alpha+1}.$$

The result then follows by energy conservation.

Lemma 4.2

Assume conditions (H1)–(H4). Let $v = v(x, t)$ be the solution of equation (15), where $h \in C_0^\infty(\mathbb{R}^3)$, $h \geqslant 0$ ($h \not\equiv 0$) and the support of h is contained in $\{|x| \leqslant R\}$. Let (x_0, t_0) be such that $|x_0| \leqslant t_0 - R$, $t_0 \geqslant R$. If $Lf(v)(x_0, t_0) \neq 0$ then there exists $\epsilon_0 = \epsilon_0(t_0) > 0$ such that the solution u of problem (24) satisfies $u(x_0, t_0, \epsilon) \neq 0$ for $0 < \epsilon \leqslant \epsilon_0$.

Proof. Suppose that there exists a sequence $\{\epsilon_n\}_{n=1}^\infty$ with $\epsilon_n > 0$. $\epsilon_n \to 0$ as $n \to \infty$ such that $u(x_0, t_0, \epsilon_n) = 0$ for all n. Since $u = v - \epsilon Lf(u)$, then by evaluating this identity in (x_0, t_0) and letting $\epsilon = \epsilon_n$ we obtain that $Lf(u)(x_0, t_0) = 0$. Iterating once it follows that $Lf(v - \epsilon_n Lf(u))(x_0, t_0) = 0$. Hence

$$Lf(v)(x_0, t_0) = L[f(v) - f(v - \epsilon_n Lf(u))](x_0, t_0). \tag{25}$$

We now take limit as n tends to infinity. By Lemma 4.1, $|Lf(u)|$ is bounded independently of ϵ_n in $\Omega(x_0, t_0)$. Therefore, by the Lebesgue's dominated convergence theorem, it follows that $Lf(v)(x_0, t_0) = 0$, which contradicts our assumption.

Theorem 4.1

Let us assume conditions (H1)–(H4). Let h be as in Lemma 4.2. Given $\rho \geqslant R$, there exists $\epsilon_1 = \epsilon_1(\rho) > 0$ such that the solution u of equation (23) does not vanish identically in the forward cone $K_\rho = \{|x| \leqslant t - \rho, t \geqslant \rho\}$ if $0 < \epsilon \leqslant \epsilon_1$.

Proof. Since $f(u) > 0$ whenever $u > 0$, then as in the proof of Theorem 3.1 we can show that $L(v)(x_0, t_0) > 0$. Thus Lemma 4.2 concludes the proof.

Corollary 4.1

Let us assume all of the conditions of Theorem 4.1. Then there exists a C^∞ solution of

$$u_{tt} - \Delta u + f(u) = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}^+$$

with C^∞ initial data with compact support, which is not identically zero in

$$\{(x, t), |x| \leqslant t - R/\sqrt{\epsilon}, t \geqslant R/\sqrt{\epsilon}\}$$

for small $\epsilon > 0$.

Proof. It follows directly from Theorem 4.1 via the scaling $u(x, t) = w(x/\sqrt{\epsilon}, t/\sqrt{\epsilon})$ where w satisfies $w_{tt} - \Delta w + \epsilon f(w) = 0$.

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